2. ARITHMETIC

§2.1. Numbers

I’ll assume that you know how to add up whole numbers, and that you know your times tables. Of course you may prefer to use a calculator and, if you have a lot of calculations to perform, there’s nothing wrong with using a calculator.

A lot of nonsense gets said about how allowing students to use calculators undermines their mathematical ability. Calculators and computer spreadsheets are wonderful tools. I’m much happier with my accountant preparing my tax return with a calculator than if he did it all in his head. The human mind is a wonderful thing but it is not as consistently accurate as a calculator.

On the other hand it’s important to be able to do calculations in one’s head if called upon to do it. You’d be annoyed if the person in the supermarket queue whisked out their calculator to check that they were being given the right change. Teachers laugh about students using a calculator to multiply some number by 10, and taking quite a few seconds to do so. (Hopefully they have enough sensitivity no to laugh in front of the student, but in the staff room teachers love to swap stories of silly things some of their students have done.)

Mental arithmetic was once a major part of basic mathematics. It has largely gone out of favour, and certainly there was probably too much importance attached to it 50 or 100 years ago. But being able to do calculations in one's head is an important skill, not just in case the battery of one's calculator goes flat.

In all of mathematics certain things need to be done in one's head – not just simple arithmetic. Mental arithmetic trains the mind by forcing you to keep partial results in your head. That ability transfers to steps in more advanced areas of mathematics. You're crippling yourself if you have to write down every single step of a calculation. Imagine if you had never learnt to walk as a single operation and always had to follow a sequence of "lift up your left leg, move it forward, put it down, lift up your right leg, ..."

Even calculators can give wrong answers if you accidentally press wrong keys, or fail to clear previous answers before you begin. It's important to sense when an answer given by a calculator is clearly wrong. It often happens that a student will add two numbers like 371 to 569 on their calculator and believe the answer 2170 that somehow pops up on the screen of their calculator. One should get into the habit of asking oneself "is this a reasonable answer?" In this case you should expect an answer in the 800s or 900s. Certainly 2170 is ridiculous.

There are methods of checking additions that are not always taught at school. One technique is to add up the numbers again, but in a different order. It's a fact of arithmetic that you get the same answer if you add numbers in any order. Merely checking the addition by repeating it in the same order won’t pick up systematic mistakes, such as a belief that 9 + 8 = 15. So, if you have a column of figures, add them from the top down and check your answer by adding them from the bottom up.

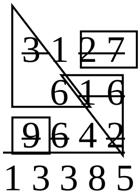
Another useful technique, when using a calculator is to write down the total and then subtract off each number until, hopefully, you get zero. If you don’t get zero you’ve made an error. Perhaps you keyed in 3197 instead of 3917.

Another technique is called “casting out 9s”. You just treat the digits as individual numbers and cross out groups that add up to 9. Having done this you add the remaining digits. Then you add the digits of this total, and so on until you get a single digit. You do the same with the total you’ve obtained and you should get the same digit. If you get different digits you’ve made a mistake somewhere. Of course it could be a mistake in your checking and not with the original answer. And, if you get the same digit, your answer could be still be incorrect, but the chance of this would be 1 in 9. Casting out groups that total 9 is not necessary, but it makes the task much easier.

Example 1:

$$\begin{array}{r} 3\,1\,2\,7 \\ 6\,1\,6 \\ \underline{\quad 9\,6\,4\,2} \\ 1\,3\,3\,8\,5 \end{array}$$

We cast out the 2 and 7, the 9, the 1, 6 and 2 and the 3 and 6 because all these groups total 9.



At this stage we don't do anything to our total.

Adding the rest we get 1 + 6 + 4 = 11.
Since 1 + 1 = 2 we end up with 2.
Casting out 9s in our total we get:
~~1~~ 3 3 ~~8~~ 5 and are left with 3 + 3 + 5, since 1 + 8 = 9.
Now 3 + 3 + 5 = 11 and 1 + 1 = 2.

The fact that we got 2 both times doesn't prove that our answer is definitely correct, but it means it is quite likely to be correct.

After a little practice this process will become second nature. Another technique for checking totals of long lists of numbers is to group them, such as groups of 5 or 8, and total each group. Then add the group totals. This is useful even when using a calculator. Although calculators can be relied on to be accurate, it's easy to enter a number incorrectly or to leave out a number altogether.

Who has never had to total a long list of numbers only to find that every time you check the total you get a different answer! By breaking the list into smaller groups it’s easier to check.

A version of this is used when data is arranged in a table. One calculates the totals of each row and of each column. You then calculate the sum of the row totals and the sum of the column totals. These should be the same.

Example 2:

3167	513	8712	469	**12861**
10823	6297	3166	8019	**28305**
46	2877	9628	1424	**13975**
14036	**9687**	**21506**	**9912**	**55141**

Subtraction is performed similarly. We proceed from the right hand end. If you find yourself having to subtract a larger digit from a smaller you ‘borrow’ a 1 as in the following example.

Example 3: Subtract 3175 from 8234.
Solution: We start by writing down
8 2 3 4 –
3 1 7 5

We can’t subtract 5 from 4 so we ‘borrow’ 1 from the next column but ‘pay it back’ as follows.
8 2 3 $^{1}4$ –
3 1 7^{1} 5

What we’re really doing is transferring a 1 from the tens column of 3175 and attaching it to the 4 in 8234 to make 14. In effect we’ve added 10 to the 8234 and added 10 to the 3175, which won’t affect the answer. It’s like giving with one hand and taking away with the other.

Now at this point many mathematics educators will be throwing up their hands in despair. Borrowing and paying back is considered to be the wrong way to teach subtraction. Children these days are taught a variety of methods which are supposed to improve the children’s understanding. But in most cases it slows them down. The perspective of a working mathematician, rather than a mathematics educator, is that understanding should follow performance.

After all, when using a calculator we don’t need to know how the calculator carries out the calculation. It doesn’t use the methods that are used when doing things by hand. Calculators convert numbers to binary and use binary arithmetic before converting them back. And, of course, if you want to really understand what is going on inside the calculator you had better know something about electronics, and semi-conductors! In the opinion of a mathematician there are more interesting, and more important things to understand in mathematics.

Let’s complete the above subtraction.

$$\begin{array}{l} 8\;2\;3\;{}^{1}4\;- \\ \underline{3\;1\;7^{1}\;5} \\ \phantom{3\;1\;7^{1}\;}9 \end{array}$$

Five from 14 is 9. We put this down. Now, moving left we have to subtract 8 (the 7 plus the paid-back 1) from 3. Again we borrow.

$8\ 2\ {}^{1}3\ {}^{1}4\ –$
$\underline{3\ 1^{1}\ 7^{1}\ 5}$
9
Eight from 13 is 5.
$8\ 2\ {}^{1}3\ {}^{1}4\ –$
$\underline{3\ 1^{1}\ 7^{1}\ 5}$
$5\ \ 9$

Now we have 2 (that is 1 + 1) from 2, and then 3 from 8. No borrowing required here. The final answer is 5059.
$8\ 2\ {}^{1}3\ {}^{1}4\ –$
$\underline{3\ 1^{1}\ 7^{1}\ 5}$
$5\ 0\ \ 5\ \ 9$

A word of warning. If you have small children don't attempt to help them with their subtraction homework unless you take the trouble to read up on how they're being taught. They'll get marked wrong, even if they get the right answer, if they even so much as hint that they're borrowing!

§2.2. Multiplication

Long multiplication and long division are procedures that are taught in primary school. We call the result of multiplying two or more numbers, their **product** and the result of dividing one number by another their **quotient**. Of course these days we use calculators to do our arithmetic, especially multiplication and division. But there are those who think it is good for your soul to learn how to do it by hand. You never know when you’ll be on a desert island without any electronic devices, and it’s a life and death matter for you to multiply 6285 by 3076. So I am obliged to show you how to do long multiplication and long division!

Example 4: Multiply 6285 by 3076.
Solution: We begin by multiplying 6285 by the 6 of 3076, getting 37710.

$$\begin{array}{r} {}^{1}6\ {}^{5}2\ {}^{3}8\ 5 \\ \underline{3\ \ 0\ \ 7\ \mathbf{6}}\ \times \\ \mathbf{3\ \ 7\ \ 7\ \ 1\ 0} \end{array}$$

The little numbers on the left of the digits of 6285 are the ‘carry digits’. We say 6 time 5 is 30, put down the 0 and carry the 3. Then we say 6 times 8 is 48, plus the 3 that was carried, making 51. Put down 1 and carry the 5. Then 6 times 2 is 12, plus the carried 5 makes 17. Put down the 7 and carry the 1. Finally we say 6 times 6 is 36, plus the carried 1 makes 37. We write down 37.

Now we multiply 6285 by the 7. Since we’re really multiplying by 70 we put a 0 in the right hand column on the next line and proceed as before. We’ve placed the carrying digits above the corresponding digits of 3076.

$$\begin{array}{rrrrrrl}
 & & {}^{1}6 & {}^{5}2 & {}^{3}8 & 5 & \\
 & & 3 & 0 & \mathbf{7} & 6 & \times \\ \cline{3-6}
 & 3 & 7 & 7 & 1 & 0 & \\
\mathbf{4} & \mathbf{3} & \mathbf{9} & \mathbf{9} & \mathbf{5} & \mathbf{0} &
\end{array}$$

Normally we'd then put down two 0's on the next line but there's no point in multiplying by the 0 of 3076 so we jump to multiplying by 3 (in effect we are multiplying by 3000). This means we put three 0's on the right of the third row.

$$\begin{array}{rrrrrrrrl}
 & & & & 6 & {}^{2}2 & {}^{1}8 & 5 & \\
 & & & & \mathbf{3} & 0 & 7 & 6 & \times \\ \cline{5-8}
 & & & 3 & 7 & 7 & 1 & 0 & \\
 & & 4 & 3 & 9 & 9 & 5 & 0 & \\
1 & 8 & 8 & 5 & 5 & 0 & 0 & 0 &
\end{array}$$

Finally we add all these products together.

$$\begin{array}{rrrrrrrrl}
 & & & & 6 & {}^{2}2 & {}^{1}8 & 5 & \\
 & & & & \mathbf{3} & 0 & 7 & 6 & \times \\ \cline{5-8}
 & & & 3 & 7 & 7 & 1 & 0 & \\
 & & 4 & 3 & 9 & 9 & 5 & 0 & \\
1 & 8 & 8 & 5 & 5 & 0 & 0 & 0 & \\ \cline{1-8}
\mathbf{1} & \mathbf{9} & \mathbf{3} & \mathbf{3} & \mathbf{2} & \mathbf{6} & \mathbf{6} & \mathbf{0} &
\end{array}$$

Of course the carrying digits change at each stage and it may be confusing to have them there all at the same time. You can simply keep them in your head, but if you insist on writing them down you may have to write them one above the other.

$$
\begin{array}{cccccccc}
 & & & & & 2 & 1 & \\
 & & & & 1 & 5 & 3 & \\
 & & & & {}^{1}6 & {}^{5}2 & {}^{3}8 & 5 \\
 & & & & \underline{\mathbf{3}} & \underline{0} & \underline{7} & \underline{6}\ \times \\
 & & & 3 & 7 & 7 & 1 & 0 \\
 & & 4 & 3 & 9 & 9 & 5 & 0 \\
\underline{1} & \underline{8} & \underline{8} & \underline{5} & \underline{5} & \underline{0} & \underline{0} & \underline{0} \\
\mathbf{1} & \mathbf{9} & \mathbf{3} & \mathbf{3} & \mathbf{2} & \mathbf{6} & \mathbf{6} & \mathbf{0}
\end{array}
$$

So the product of 6285 and 3076 is 19,332,660. Confused? With all this carrying we'd get all these little carrying numbers dotted over our work lie dead flies.

Well, here's an alternative method of long multiplication that you would never have come across at school. See if you like it any better.

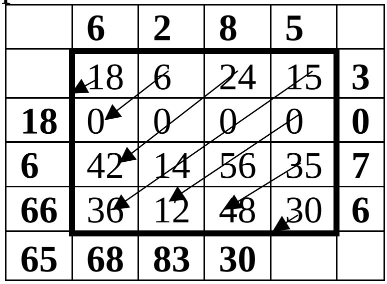
Example 4 (again): Multiply 6285 by 3076.
Solution: Set up a table where the digits of 6285 go across the top and the digits of 3076 go down the right hand side.

	6	**2**	**8**	**5**	
					3
					0
					7
					6

Now multiply the digits, putting the products in the appropriate cells.

	6	**2**	**8**	**5**	
	18	6	24	15	**3**
	0	0	0	0	**0**
	42	14	56	35	**7**
	36	12	48	30	**6**

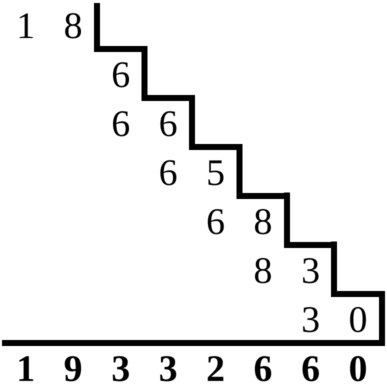
Now add the diagonals in the direction and put these in the appropriate places in the left column and bottom row.



	6	**2**	**8**	**5**	
	18	6	24	15	**3**
18	0	0	0	0	**0**
6	42	14	56	35	**7**
66	36	12	48	30	**6**
65	**68**	**83**	**30**		

Note that the diagonals correspond to numbers that really should be followed by the same number of 0's. So the 24 and the 42 both represent 240000 and 420000 and so the total 66 represents 660000. And the 35 and 48 represent 350 and 480 respectively and so their total, 83, represents 830.

Finally, draw a staircase and write these numbers to the left of the verticals, starting with the top total (18 in this case) and going down, and around across the bottom. Then total these, carrying where necessary.



This method reduces the carrying to the extent that we could keep the carried digits in our head. But if you’re used to the standard method, by all means continue to use it on the rare occasions you can’t use a calculator.

§2.3. Division

As I said above, these days if you want to multiply or divide you use a calculator. Yet you are supposed to be able to do it without such aids.

Many years ago, as a student, I had a vacation job doing statistics. What joy when I realised that I would be using an electric Facit calculator. At university in statistics workshops we used mechanical versions, where we had to turn a handle, Division was particularly laborious as we had to turn the handle many times. But with this version you entered the two numbers and pressed the DIVIDE button.

Division was done by repeated subtraction, If we wanted to divide 7738 by 23 the machine would automatically subtract repeatedly from 77 until the answer became negative, then rotated back once. This would result in the 7738 being replaced by 838 (the 8 being the remainder on dividing 77 by 23 and the 38 unchanged) and 3 (the number of times the 23 was subtracted) appeared in the answer register.

Then the register holding the 23 moved one place to the right and the process automatically repeated, this time with the 23 being subtracted repeatedly from 83.

All this happened once the DIVIDE key was pressed. I just sat back and waited for the machine to stop, which might take as much as twenty seconds. You are used to electronic calculators where the answer appears instantaneously.

We thought it was wonderful. At university we had to turn a handle to do the repeated subtraction and we had to listen out for the bell that told us that the answer had gone negative, which meant we had to turn the handle one turn back the other way. Then we had to slide a knob to the right to move the second number across on place. With these new machines we just sat back and made ourselves a cup of coffee while we waited!

There were a couple of other university students on secondment, and we used to have competitions. Who could devise a division problem that would make their machine run longest? We'd load our numbers into our respective machines and then, at the same moment, pressed our DIVIDE keys while we sat back to see whose machine would be last to finish.

One advantage of the electromechanical version over our modern electronic calculators is that it reinforces the method needed when doing it with pen and paper – the process of repeated subtraction. The following is the standard method of long multiplication. I haven't yet thought of a simpler way of doing it.

Example 5: Divide 7738 by 23.

Solution: We start by writing down

23)7738

We divide 23 into 77 getting a quotient of 3.

We write the 3 above the second 7 in 77.

$$\begin{array}{rcccc}
 & & \underline{3} & & \\
23) & 7 & 7 & 3 & 8
\end{array}$$

Then we multiply 23 by 3, getting 69 and write it down under the 77 and subtract.

$$\begin{array}{rcccc}
 & \underline{} & \underline{3} & & \\
23) & 7 & 7 & 3 & 8 \\
 & \underline{6} & \underline{9} & \downarrow & \\
 & & 8 & 3 &
\end{array}$$

We brought down the 3, giving 83.

We now divide 83 by 23. Since 3 times 23 is 69 and 4 times 23 is 92 we must go with a quotient of 3. We continue in this way.

$$\begin{array}{rcccc}
 & \underline{} & \underline{3} & \underline{3} & \underline{6} \\
23) & 7 & 7 & 3 & 8 \\
 & \underline{6} & \underline{9} & & \big| \\
 & & 8 & 3 & \big| \\
 & & \underline{6} & \underline{9} & \downarrow \\
 & & 1 & 4 & 8 \\
 & & \underline{1} & \underline{3} & \underline{8} \\
 & & & 1 & 0
\end{array}$$

We can report this as 336, with a remainder of 10, or as an improper fraction as $336\frac{10}{23}$. But we'll discuss fractions in the next section.

Once again don't attempt to help your children with these methods. Mathematics educators regard 'carrying' as old-fashioned as 'borrowing' and have their own explanations as to how to perform these operations. That's fine, but remember that the important thing is to be able to perform multiplication and division quickly and accurately. Often these "modern methods" can slow you down. The kids of today have to humour their teachers but you can choose whichever methodology is quickest and easiest!

Divisibility is another skill that you need to look at. One number is **divisible by** another if it goes into it exactly, that is, with no remainder. So 2 divides 18 and 7 divides 21.

Numbers that are divisible by 2 are called **even**, and the others are called **odd**.

Now some teachers, and even some school textbooks, get confused when it comes to zero. Is 0 odd or even? Of course it's even. An even number is double a whole number and twice zero is zero. Yet would you believe that some teachers are adamant that "zero is neither odd nor even"? They're sure they're correct because they've seen it in print! I think where this confusion arises is the fact that 0 is neither positive nor negative and they mistakenly extend this to odd-ness and even-ness. If you have a child who comes home believing that 0 is neither odd nor even just let it ride! Many teachers are so firmly entrenched in this fallacy that they get angry when someone challenges it. It’s not worth the effort!

Now the test as to whether a number is even is simply to see whether it ends in an even number. So 1234567894 is even. You don’t need to start dividing by 2. Any number ending in 0, 2, 4, 6 or 8 is even – the others are odd. (Even 0 conforms to this pattern. It ends in 0 and so it is even.)

Divisibility by 5 and 10 are also easy. A number is divisible by 10 precisely when it ends in 0 and it is divisible by 5 if it ends in 0 or 5.

Divisibility by 3 and 9 are almost as easy to test. You add up the digits and if the total of the digits is divisible by 3 so is the number – otherwise it is not. Similarly if the digit total is divisible by 9, so is the number.

Example 6: Which of the numbers 148272 and 111111111 are divisible by 3? Which are divisible by 9?
Solution: The digit total for 148272 is 24. This is divisible by 3 and so 148272 is divisible by 3. But 24 is not divisible by 9 and so 148272 is not divisible by 9.
The digit total of 111111111 is 9 and so the number is divisible by 9, and hence by 3.

DIVISIBILITY TESTS

DIVISIBILE BY	**TEST**
2	ends in 0, 2, 4, 6 or 8
3	digit sum is divisible by 3
4	last two digits give a multiple of 4
5	ends in 0 or 5
9	digit sum is divisible by 9
10	ends in 0

There are divisibility tests for other numbers but they’re rather more complicated and are not worth the effort learning them. There have actually been whole books published on the subject, including tests for divisibility by 7, 11, 13 and even 23. The authors believe they are making a real contribution to mathematics. Take it from a mathematician that there’s no need to learn any divisibility tests beyond those above.

§2.4. Fractions

We’ll dispense with all the usual stuff about cutting cakes into so many equal slices. Even innumerate people know what a fraction is. And people meet fractions in places other than sharing cakes. Mostly fractions occur as proportions. One third of all marriages end in divorce. Three quarters of all married couples own their own home, or are in the process of paying off a mortgage.

It is doing arithmetic where some people come to grief. What is $\frac{1}{2}+\frac{1}{3}$? There are some people who believe the answer is $\frac{2}{5}$. Just add the tops and add the bottoms and, there you have it, $\frac{2}{5}$. This is a case of people inventing their own rules. It seems to be a very natural thing to do, so it must be correct, but it isn't!

The problem with fractions is the fact that different looking fractions can represent the same quantity. After all, $\frac{3}{4}$ is equal to $\frac{6}{8}$, or $\frac{30}{40}$, or $\frac{75}{100}$. You can multiply or divide the **numerator** and **denominator** of a fraction (these are just fancy names of the top and the bottom) by the same number and what you get is an **equivalent fraction**, that is, a fraction whose value is equal to the one you started with.

Often we like to **reduce a fraction to lowest terms**. This means dividing numerator and denominator by common factors until it is no longer possible. Dividing numerator and denominator by some number is called **cancelling**.

Example 7: Reduce $\frac{18}{27}$ to lowest terms.

Solution: We can cancel by 3 to get $\frac{18}{27} = \frac{6}{9}$. But this isn't in lowest terms because we can cancel again by 3 to get $\frac{6}{9}$

$= \frac{2}{3}$. Of course we would probably do this in one go by cancelling by 9.

A **proper fraction** is one where the numerator is less than the denominator, such as $\frac{3}{5}$ or $\frac{19}{20}$. An **improper fraction** is where the numerator is equal to, or larger than, the denominator, such as $\frac{5}{4}$ or $\frac{22}{7}$. An improper fraction can be converted to a whole number or a **mixed fraction** (one that is a mixture of a whole number and a proper fraction) by dividing the denominator into the numerator, getting a quotient and a remainder.

Example 8: Write $\frac{7}{4}$, $\frac{22}{7}$ and $\frac{75}{25}$ as improper fractions.

Solution: $\frac{7}{4} = 1\frac{3}{4}$. If we divide 4 into 5 we get a quotient of 1 and a remainder of 3.

$\frac{22}{7} = 3\frac{1}{7}$ because 7 goes into 22 three times, leaving a remainder of 1 (22 – 3 × 7 = 1).

$\frac{75}{25} = 3$. Here the remainder is zero.

In daily life we usually prefer mixed fractions to improper ones. A quantity of $\frac{9713}{4}$ doesn’t convey as much as $2428\frac{1}{4}$. But when doing arithmetic with fractions we always use improper fractions. So, we need to be able to convert mixed fractions to improper fractions and vice versa.

Example 9: Convert $10\frac{5}{7}$ to an improper fraction.

Solution: $10\frac{5}{7} = \frac{75}{7}$. Here the whole number, 10, becomes 70 sevenths. Then we add the 5 sevenths to get 75 sevenths.

§2.5. Addition and Subtraction of Fractions

So let us learn how to add and subtract fractions. To begin with we will add two fractions with the same denominator. In this case you simply add or subtract the numerators.

Example 10: $\frac{13}{17} + \frac{2}{17} = \frac{15}{17}$ and $\frac{13}{17} - \frac{2}{17} = \frac{11}{17}$.

Just think of seventeenths like oranges. 13 oranges plus 2 oranges make 15 oranges. The denominator doesn’t change. But sometimes, once you get the answer. you can cancel and so simplify.

Example 11: $\frac{7}{18} + \frac{5}{18} = \frac{12}{18} = \frac{2}{3}$ and $\frac{7}{18} - \frac{5}{18} = \frac{2}{18} = \frac{1}{9}$.

The tricky bit is when we have to add two fractions with different denominators. The first step is to change them so that they have a common denominator. This means finding a number that both denominators divide.

Example 12: Find $\frac{7}{12} + \frac{5}{8}$.

Solution: Here a suitable common denominator is 24. We now ask what you need to multiply 12 by to get 24. The answer is 2. So we must multiply the 7 by 2 to get $\frac{7}{12} = \frac{14}{24}$.

Similarly we need to multiply 8 by 3 to get 24 so we multiply the 5 by 3 to get $\frac{5}{8} = \frac{15}{24}$.

So $\frac{7}{12} + \frac{5}{8} = \frac{14}{24} + \frac{15}{24} = \frac{29}{24}$. We could write this as $1\frac{5}{24}$ but usually it is best to leave the answer as an improper fraction.

Sometimes we cannot easily think of a suitable common denominator. If the worst comes to the worse you can always use the product of the denominators as a common denominator but sometimes this involves more arithmetic.

Example 12 (again): Find $\frac{7}{12} + \frac{5}{8}$.

Solution: We could take 8 times 12 as a common denominator. We would then have

$\frac{7}{12} + \frac{5}{8} = \frac{56}{96} + \frac{60}{96} = \frac{116}{96} = \frac{29}{24}$ after cancelling by 4.

A good way of getting the *least* common denominator, and thereby minimising the arithmetic, is to set up tables of multiples of the two denominators.

Example 13: Find $\frac{7}{24} + \frac{13}{60}$.

Solution: Repeatedly adding 24 and 60 we get

1	24	60
2	48	**120**
3	72	
4	96	
5	**120**	

So 120 is the least common multiple. We have to multiply top and bottom of the first fraction by 5 and the second fraction by 2.

Hence $\frac{7}{24} + \frac{13}{60} = \frac{35}{120} + \frac{26}{120} = \frac{61}{120}$.

§2.6. Multiplication and Division of Fractions

In these cases we do not need a common denominator. To multiply fractions we simply multiply the numerators and multiply the denominators.

Example 14: $\frac{3}{5} \times \frac{2}{7} = \frac{6}{35}$.

When we get the answer it is always good to simplify by cancelling common divisors. This can often be made easier by cancelling before we multiply.

Example 15: Find $\frac{3}{5} \times \frac{25}{27}$.

Solution: We certainly can multiply top and bottom to get $\frac{75}{135}$ and then simplify by cancelling. But it’s much easier to cancel by 3 and by 5 first.

$$\frac{3}{5} \times \frac{25}{27} = \frac{\cancel{3}}{\cancel{5}} \times \frac{\cancel{25}^{5}}{\cancel{27}^{9}} = \frac{5}{9}.$$

To divide fractions you simply **invert and multiply**. That is, you turn the second fraction upside down and then multiply.

Example 16: Find $\frac{6}{35} \div \frac{27}{140}$.

Solution: We write $\frac{6}{35} \div \frac{27}{140} = \frac{6}{35} \times \frac{140}{27}$

$$= \frac{\cancel{6}^{2}}{\cancel{35}^{5}} \times \frac{\cancel{140}^{20}}{\cancel{27}^{9}}$$

$$= \frac{\cancel{6}^{2}}{\cancel{35}^{\cancel{5}}} \times \frac{\cancel{140}^{\cancel{20}\ 4}}{\cancel{27}^{9}} = \frac{8}{9}.$$

§2.7. Decimals

Many years ago, in Australia and Britain, we used pounds, shillings and pence. Distances were measured in miles, yards, feet and inches. The relationships between these units were complex. Twenty shillings make one pound, twelve pence make a shilling. A mile is 1760 yards, a yard is three feet and twelve inches make up one foot. There was little need for decimals. If you want to make a measurement more accurate you just used smaller units. Of course arithmetic involving these units was very messy. Imagine multiplying 16 pounds, 13 shillings and 6 pence by seven!

But you will be familiar with the metric system and decimal currency. There is no need for a mixture of units. You just use one unit and use decimals to get greater precision.

So an amount of $36.79 means 36 dollars, 7 tenths of a dollar and 9 hundredths of a dollar, or 36 dollars and 79 cents. A length of 9.875 metres is 9 metres plus 8 tenths of a metre, 7 hundredths of a metre and 5 thousands of a metre. We might use other units, such as 987.5 centimetres or 9875 millimetres but we never need to use more than one unit at a time.

So 29.387 means 29 plus 3 tenths plus 8 hundredths plus 7 thousandths. Where numbers are less than 1 we put a zero in front of the decimal point, so we usually write 0.123 rather than .123. This represents one tenth plus two hundredths plus 3 thousandths.

Even smaller numbers will have zeros immediately after the decimal point. So the decimal 0.00123 represents one thousandth, two ten thousandths plus 3 hundred thousandths.

Doing arithmetic with decimals is easy in most cases. To add or subtract two decimals you write them underneath one another **so that the decimal points line up**. Then you add or subtract in the usual way, treating any blanks as zeros. The decimal point in the answer goes underneath the others.

Example 17: Add 6.9, 562.83 and 15.3376.

Solution:

6.9
562.83
15.3376
585.0676

If it helps you can replace the blanks after the decimal point by 0’s but normally we don’t.

6.9000
562.8300
15.3376
585.0676

Example 18: Subtract 6.8275 from 19.2.

Solution:

19.2000 –
6.8275
12.3725

When it comes to multiplying or dividing you perform the calculation as with whole numbers and then work out where to put the decimal point in the answer by adding or subtracting the number of decimal places in the original two decimals. You leave out the decimal point until you get the final answer.

If you are multiplying you add the numbers of decimal places. If you are dividing you subtract the numbers of decimal places.

Example 19: Multiply 18.24 by 3.123.

Solution: 1824

$$\begin{array}{r} \underline{3123 \times} \\ 5472 \\ 36480 \\ 182400 \\ \underline{5472000} \\ 5696352 \end{array}$$

There are 2 decimal places in 18.24 and 3 decimal places in 3.123 so there are 5 decimal places in the answer. The answer is therefore 56.96352.

Example 20: Divide 78.244 by 3.1.

Solution:

$$\begin{array}{l} \qquad\ \ \underline{2\ 5\ 2\ 4} \\ 31)7\ 8\ 2\ 4\ 4 \\ \qquad \underline{6\ 2} \\ \qquad 1\ 6\ 2\ 4\ 4 \end{array}$$

1 6 2 4 4 (repeated from the previous page)
1 5 5
7 4 4
6 2
1 2 4
1 2 4
0

There are 3 decimal places in 78.244 and 1 decimal place in 3.1 so there are 2 decimal places in the answer. The answer is therefore 25.24. Here we get a remainder of 0 and so get an exact solution. But frequently this doesn’t happen, and if we want more decimal places we continue, bringing down 0’s.

Example 21: Divide 1 by 3. Continue forever.
Solution:

3 3 3
3)1.0 0 0
9
1 0
9
1 0
9
1

Although we can’t actually go on forever we can see that if we did we’d get 0.3333333333333333333333333 ....
We call this “point three repeater” and write it as $0.\dot{3}$

So $\frac{1}{3} = 0.\dot{3}$

Example 22: Write $\frac{3}{7}$ as a decimal.

Solution:

$$\begin{array}{r}
\underline{\quad 4\,2\,8\,5\,7\,1} \\
7)3\,0\,0\,0\,0\,0\,0
\end{array}$$

$\underline{2\,8}$
2 0
$\underline{1\,4}$
6 0
$\underline{5\,6}$
4 0
4 0 repeated from the previous page
$\underline{3\,5}$
5 0
$\underline{4\,9}$
1 0
$\underline{7}$
3 0

Clearly everything will repeat from this point on.

So $\frac{3}{7}$ = 0.428571428571428571428571 ....

Here the decimal repeats in blocks. We can write this as $0.\dot{4}2857\dot{1}$

The first dot goes over the beginning of the repeated block and the second dot goes over the end of the block.

Repeating decimals needn’t begin with the repeating block.

For example $\frac{21149}{9900} = 2.1362626262\ldots\ldots = 2.13\dot{6}\dot{2}$

§2.8. Percentages

Decimals are a convenient way of representing fractions but when they represent proportions, where the fractions are less than 1, the decimals all start with 0. which seems a bit clumsy. We might hear that the proportion of voters voting for a certain party is 0.43. It is much easier to write this as 43%. Percentages are just numbers that have been multiplied by 100 for convenience. Often we don’t need decimal points in percentages because the two figures give sufficient accuracy. When we quote mortgage rates the figures are a lot less than 1 and we might be paying 5.75%. But this is still much less clumsy than quoting it as the proportion 0.0575.

To convert from decimals to percentages you just multiply by 100, which amounts to shifting the decimal point two places to the right. To convert percentages to decimals you divide by 100 which amounts to shifting the decimal point two places to the left. If necessary you may need to put extra zeros on the left. And, remember, a whole number doesn’t appear to have a decimal point, but it is there, invisibly, at the right hand end.

Example 23: Convert 0.56 and 0.0235 to a percentages.
Solution: 0.56 = 56% and 0.0235 = 2.35%/

Example 24: Convert 37.6% and 1.3% to decimals.
Solution: 37.6% = 0.376 and 1.3% = 0.013.

Percentages are mostly used where the number is less than 1 and so the percentage is normally less than 100%. But they can go higher. A percentage of 750% corresponds to the decimal 7.5.

§2.9. Powers and Roots

The **square** of a number is that number multiplied by itself. We write the square by writing a little 2 to the right of the number and up a bit. This is called a superscript. So 3^2 is three squared, which is 9. Since you know your multiplication table, at least up to 10 × 10. you already know the squares up to 10 squared.

number	1	2	3	4	5	6	7	8	9	10
square	1	4	9	16	25	36	49	64	81	100

The **cube** of a number is that number times itself times itself, that is three copies of the number multiplied together. The cube of a number is written using a little 3 as a superscript, so that 2^3 is $2 \times 2 \times 2 = 8$. It is useful to remember a few cubes, say up to 5^3 because they often crop up in mathematics.

number	1	2	3	4	5
cube	1	8	27	64	125

Square roots are the reverse process to squaring, just as halving is the reverse process to doubling. The **square root** of a number is that number whose square is the original number. Since 3 squared is 9, the square root of 9 is 3. A perfect square is a number whose square root is an exact whole number.

perfect square	1	4	9	16	25	36	49	64	81	100
square root	1	2	3	4	5	6	7	8	9	10

We write the square root of a number using the symbol $\sqrt{\quad}$. The number goes underneath the square root symbol. So $\sqrt{9} = 3$. Now you may wonder what happens to numbers that are not perfect squares, such as 2. What is $\sqrt{2}$?

It cannot be written as a fraction and can only be written as a decimal exactly if we allow there to be infinitely decimal places.

$\sqrt{2}$ = 1.41421356237309504880168872420 ....

Of course this is only an approximation, though it is extremely close to the actual value. Notice that there is no apparent repetition, though we might suspect that it might eventually repeat if only we continued a lot further.

In fact it can be shown that the decimal expansion of $\sqrt{2}$ is not a repeating decimal. Also it is not an exact fraction. In fact it can be shown that all repeating decimals can be expressed as exact fractions.

So there are three types of number.

Type	**As a decimal**	**Example**
Whole number	No decimal point	3
Fraction	Repeating	$2.\dot{1}6\dot{7}$
Irrational	Not repeating	$\sqrt{2}$

A **cube root** of a number is a number whose cube is the original number, so the cube root of 27 is 3. We write cube roots in a similar way to square roots, but with a little 3. So the cube root of 2 is written $\sqrt[3]{2}$. Since 2 is not a perfect cube $\sqrt[3]{2}$ is irrational and so its decimal representation is non repeating. We can have fourth roots, fifth roots etc and these are treated similarly.

Numbers can built up from square roots, cube roots etc and these are called **surds**. Often surds can be simplified.

§2.10. Simplifying of Surds

Square roots and sums do not go well together. The square root of a sum is not the sum of the square roots. For example $\sqrt{9+16}$ is not $\sqrt{9}+\sqrt{16}$, which is 3 + 4 = 7. No, $\sqrt{9+16}=\sqrt{25}=5$. The same thing goes for differences.

But square roots and products work well together. The square root of a product is the product of the square roots. For example $\sqrt{4\times 9}=\sqrt{4}\times\sqrt{9}$. This can be useful in simplifying square roots.

Example 25: Simplify $\sqrt{45}$.
Solution: $\sqrt{45}=\sqrt{9\times 5}=\sqrt{9}\times\sqrt{5}=3\times\sqrt{5}$, which we write as $3\sqrt{5}$.

Example 26: Simplify $\sqrt{45}-\sqrt{20}$.
Solution: $\sqrt{45}-\sqrt{20}=3\sqrt{5}-2\sqrt{5}=\sqrt{5}$.
Example 27: Simplify $\frac{1}{\sqrt{2}}$.
Solution: Multiply numerator and denominator by $\sqrt{2}$.
Then $\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{\sqrt{2}\times\sqrt{2}}=\frac{\sqrt{2}}{2}$.
Why do we consider $\frac{\sqrt{2}}{2}$ to be simpler than $\frac{1}{\sqrt{2}}$? Both are easy to evaluate with a calculator. But we may happen to know that $\sqrt{2}$ is about 1.4. Working out $\frac{1}{1.4}$ without a calculator isn't easy but working out $\frac{1.4}{2}$ is. So $\frac{1}{\sqrt{2}}$ is approximately 0.7.

Example 28: Simplify $\frac{1}{\sqrt{5}}+\sqrt{45}$.

Solution: $\frac{1}{\sqrt{5}} + \sqrt{45} = \frac{\sqrt{5}}{5} + 3\sqrt{5} = \frac{16}{5}\sqrt{5}$.

Square roots also go well with division. The square root of a **quotient** (one number divided by another) is the quotient of the square roots.

Example 29: Simplify $\frac{\sqrt{27}}{\sqrt{75}}$.

Solution: $\frac{\sqrt{27}}{\sqrt{75}} = \sqrt{\frac{27}{75}} = \sqrt{\frac{9}{25}} = \frac{\sqrt{9}}{\sqrt{25}} = \frac{3}{5}$.

Alternatively we could write $\frac{\sqrt{27}}{\sqrt{75}} = \frac{3\sqrt{3}}{5\sqrt{3}} = \frac{3}{5}$.

Cube roots, and higher roots, can be simplified in a similar way.

Example 30: Simplify $\sqrt[3]{80}$.

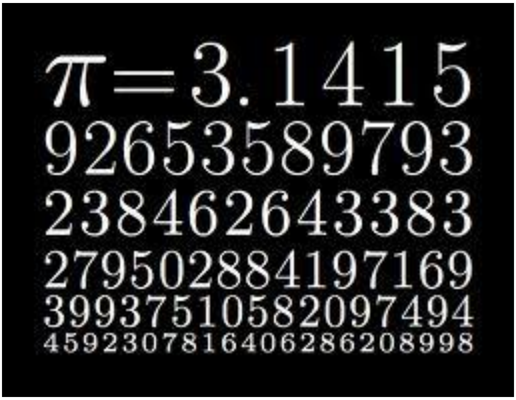
Solution: $\sqrt[3]{80} = \sqrt[3]{8\times 10} = \sqrt[3]{8}\times\sqrt[3]{10} = 2\sqrt[3]{10}$.

§2.11. The Number π

There are other irrational numbers other than surds. Like surds their decimal expansion is non repeating. The most important of these numbers in elementary mathematics is **pi**, which we write using the Greek letter π that corresponds to our letter “p”.

It was long known that the **circumference** of a circle (the distance around the edge) is a fixed multiple of the **diameter** (the distance across the circle). If you double the diameter you double the circumference. The circumference divided by the diameter is about 3. In the Old Testament (I Kings 7:23) there is a reference to a circular vessel being 30 cubits round and 10 cubits across. Of course this was only given as a convenient approximation and even back then much better approximations were known.

Some commentators have gone back to the original Hebrew, where the letters have numerical equivalents, and argue that the value of 333/106 can be inferred, which is quite accurate. At school we learnt that π is 22/7, but a careful teacher should have explained that this is only a good approximation. A better approximation is 355/113.



$\pi = 3.1415$
92653589793
238462643383
279502884197169
39937510582097494
4592307816406286208998

The fact that π cannot be expressed exactly as a fraction, that is π is irrational, means that if we want to write it down exactly we must resort to using the special symbol. So we write represent it by the special symbol π.

π = 3.14159265358979323846264338327……………… This decimal expansion is non repeating, and has no discernible pattern.

A book, which became an excellent film, is *The Life of Pi*. If you have seen it you will know that is not really about mathematics. The main character was named after an uncle who was an excellent swimmer, who trained in a swimming pool that contained the word "piscine", being the French word for swimming pool. Young Piscine, as he was called, got ragged at school for such a name and so he shortened it to Pi. But he learnt to memorise the value of π to hundreds of decimal places, which won him the admiration of his classmates, and confirmed the name Pi.

The number π has been computed to over 5 trillion decimal places and no pattern has been detected. Certainly there is no sign of it repeating. But even 5 trillion decimal places cannot answer the question as to whether the expansion eventually repeats.

This is a good example to show that although computers are an extremely useful tool in mathematics, they will never replace mathematics. A computer, spewing out the digits of π, can never, in finite time, answer the question “do the digits of π repeat?” Yet a theorem in mathematics *proves* that the answer is “no”.

§2.12. Negative Numbers

The first use for numbers was counting, and when you are counting there is little need for the number zero. True, we could say that “the number of mermaids in the universe is 0” but it would be so much easier to say that “mermaids don’t exist”.

In fact the number zero wasn’t introduced into mathematics until the early middle ages. Some say that it was an invention of the Persians (from modern day Iran). Others say that it came to Persia from India, and only then did the Europeans adopt it. It’s purpose was as a place filler in the decimal notation for numbers.

Imagine having to do arithmetic using Roman Numerals! This was a complicated system invented by the Romans where I denotes 1, V denotes 5, etc.

Roman Numeral Symbol	**Modern Day Equivalent**
I	1
V	5
X	10
L	50
C	100
D	500
M	1000

A number is built up by combining these, so that 3 is III and 200 is CC and 213 is CCXIII. But rather than writing 4 as IIII or 400 as CCCC you alter the usual order. So 4 is usually written as IV (sometimes IIII is used) and 400 is CD. A smaller symbol before a larger symbol represents the larger minus the smaller.

This is never done with 3, or 30 or 300. These are written using three symbols. So 1, 2, 3, 4, 5 are written as I, II, III, IV, V. Then come 6, 7 and 8 written as VI, VII and VIII. You have probably seen King Henry the 8th written as Henry VIII. When we come to 9 we use the "one less" principle and write it as IX. Similarly 90 is written as XC and 900 as CM. You might expect 99 could be written as IC, but it is always written as XCIX. Each digit in our system is treated separately.

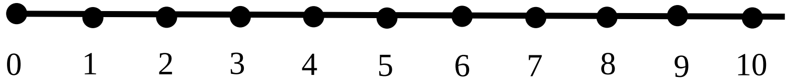
It is a clumsy system, not suitable for performing arithmetic. Try multiplying MCMXLVII by DCCXLVIII, or even worse, dividing one by the other. [In decimal notation these numbers are 1947 and 798 respectively.]

Our modern numeration system is a place value system. When we write 1947 we mean one thousand, nine hundred, four tens and seven units. What delayed the introduction of this system was the lack of a symbol for zero. If the number was one thousand, nine hundred and seven we might try to write it as 19 7. But spaces are difficult to see, especially if there are several spaces in a row.

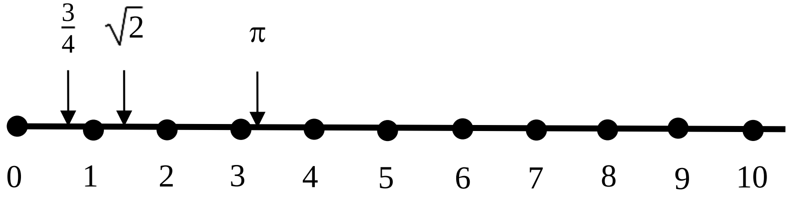
At one stage a dot was used to represent a space. So we could write it as 19.7 (of course decimal points weren’t around then). Even when the symbol 0 began to be used it was thought of as a piece of punctuation rather than as a number in its own right, rather like we use the dot in decimal notation.

Today zero is fully accepted as a number and so when we count down from 10, as one does when launching a rocket, we say 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0. But why stop there? Can’t we keep going, introducing new numbers?

After decimal numbers came into use numbers could be seen as representing points on a line. This line is called the **number line**.

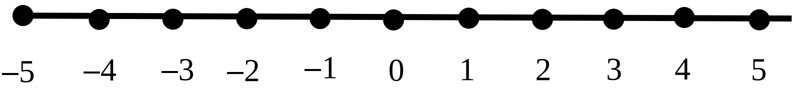


Fractions, and irrational number like $\frac{3}{4}$, $\sqrt{2}$ and π fit in between.



Now it is very natural to ask what happens to the left of zero. Subtracting 1 takes you 1 unit to the left on the number line. Why can’t you take one unit to the left of zero? Independently of this accountants found the need to express debts as well as assets. Negative numbers are the ideal model.

So one unit to the left we put the number minus 1, written –1. One unit to the left of –1 comes minus 2. So counting down from 10 we now get
10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, –1, –2, –3, ...



So if we have $2000 in the bank and we write a cheque for $3000, and if the bank honours the cheque and doesn’t bounce it, our balance will be –$1000, that is, we are overdrawn by $1000. If we then deposit $5000 our balance will be $4000.

This is reflected by the arithmetical statements (we’ll dispense with the thousands):

$$2 - 3 = -1$$

$$-1 + 5 = 4.$$

Modern day accountants haven’t quite caught up with the innovation of negative numbers. After all, they have only been around for four hundred years! They write negative balances by enclosing the number in parentheses. So on a balance sheet a balance of ($7000) means a debt of $7000, or as mathematicians would write it, –$7000.

To subtract a larger number from a smaller you instead subtract the smaller from the larger and place a minus sign in front. To add a negative number to a positive number you subtract it. To add two negative numbers you just add the numbers without the minus signs and then place a minus sign in front of your answer.

Example 31:

(i) $2 - 9 = -7$

(ii) $-8 + 11 = 3$

(iii) $-3 - 2 = -5$.

Multiplying and dividing with negative numbers cannot be explained by profit and loss, or by walking left or right along the number line. A justification for the following rules can be given but at this stage it would only confuse you. Just accept the following table.

+	**times**	+	=	+
+		–		–
–		+		–
–		–		+

Example 32:

(i) $-6 \times 2 = -12$

(ii) $-3 \times -4 = 12$.

§2.13. Complex Numbers

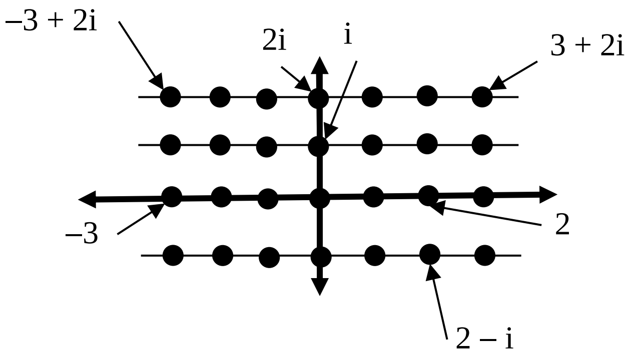
All the numbers we have considered so far are called **real numbers**. There are other numbers, though you won't need to use them unless you go much further with mathematics. So you can ignore this section if you wish.

From the above set of rules you can see that if you square any number, except zero, you get a positive number (one that is bigger than zero). Plus times plus is plus and minus times minus is plus. So $\sqrt{-1}$ does not exist. But mathematicians have always worked on the assumption that if something doesn't exist you should consider inventing it.

So we invent a new number, that we denote by the symbol **i** that has the property that $i^2 = -1$. The symbol "i" stands for "imaginary". It was considered that this number doesn't really exist but it was found that having such imaginary numbers could be extremely useful. These days they are considered to be no more than our so-called real numbers. In a sense all numbers are imaginary – they exist only in the human mind. We call a number **imaginary** if it is a real multiple of i, such as 3i or –2i.

A **complex number** is one that is the sum of a real number and an imaginary number, such as, 3 + 2i. Don't expect us to be able to simplify this. This is as simple as 3 + 2i can become, just as $\frac{3}{4}$ is as simple as we can make it. The term “complex” doesn’t mean “complicated” – just simply that it is made up of two parts.

But where can these numbers be placed on the number line? The number is completely filled by all the real numbers. The answer is simply to go into a second dimension.



What about going into three dimensions? The answer is that you don’t get a particularly useful number system by going beyond two dimensions. Most of mathematics, even very advanced mathematics, uses these two-dimensional numbers that are called “complex numbers”.